



COMSATS Institute of Information Technology, Islamabad
Department of Mathematics
Assignment # 3

Program: BSM

Instructor: Dr. Muhammad Saeed Akram

Semester: VI

Maximum Marks: 10

Date: Oct 28, 2015

Deadline : November 4, 2015

Assignment Topics: Uniform convergence of sequence and series of functions, Uniform convergence and continuity, uniform convergence and Integration, uniform convergence and Differentiation, Stone Weierstrass Theorem.

Attempt the following:

1. Let $\{f_n\}$ be a sequence of function on E , which converges pointwise on E . Then prove that the pointwise limit, f , of the sequence is unique. What can be said about uniform convergence?
2. Let $\{f_n\}$ and $\{g_n\}$ be two sequence of functions defined and pointwise convergent on a common domain E . Then prove that $\{f_n \pm g_n\}$, $\{f_n \cdot g_n\}$ and $\{|f_n|\}$ are pointwise convergent on E . What can be said about uniform convergence?
3. Let $\{f_n\}$ be a sequence of functions on E , and suppose that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. If for each $x \in E$, there exists $K_x \in \mathbb{R}$ such that $|f_n(x)| < K_x$, then prove that $\{f_n\}$ converges pointwise on E .
4. Let $\{f_n\}$ be a monotone increasing sequence of function on $[a, b]$. If each f_n is bounded on $[a, b]$, then prove that the uniform limit, f is bounded on $[a, b]$.
5. For any real $x \neq 2m\pi$ (m is an integer), we have

$$\sum_{k=1}^n e^{ikx} = e^{ix} \frac{1 - e^{inx}}{1 - e^{ix}} = \frac{\sin(nx/2)}{\sin(x/2)} e^{i(n+1)x/2}.$$

This identity yields the following estimate:

$$\left| \sum_{k=1}^n e^{ikx} \right| \leq \frac{1}{|\sin(x/2)|}.$$

6. Discuss the uniform convergence of the series

i $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, $p \in (0, \infty)$ and $x \in [a, b]$ such that $0 < a < b < 2\pi$.

ii $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$, $p \in (0, \infty)$ and $x \in [a, b]$ such that $0 < a < b < 2\pi$.

7. Let X be a metric space. Prove that $\mathcal{C}(X)$ is a metric space with the metric defined by

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

8. Show that $(\mathcal{C}(X), d)$ is a complete metric space.

9. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
10. Can the corollary after Theorem 7.15 be used to evaluate
- $\int_{-1}^1 \ln(1+x) dx$
Hint: use the fact that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, $|x| < 1$.
 - $\int_a^b e^{x^2} dx$.
 - $\int \cos x dx = \sin x$.
 - $\int_{-1}^1 \tan^{-1} x dx$.
Hint: use the fact that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$, $|x| < 1$.
11. State analogue of Theorem 7.17 for series of functions. Can it be used to show that
- if $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $x \in (0, 1)$, then $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$, $x \in (0, 1)$.
 - $\frac{d}{dx}(e^x) = e^x$, $x \in \mathbb{R}$.
12. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x) x^n dx = 0 (n = 0, 1, 2, \dots),$$

prove that $f(x) = 0$ on $[0, 1]$.

Hint: The integral of the product of f with any polynomial is zero. Use the Stone Weierstrass theorem to show that $\int_0^1 f^2(x) dx = 0$.