



COMSATS Institute of Information Technology, Islamabad
Department of Mathematics
Excercise Set 1

Program: MS/PhD
Semester:

Instructor: Dr. M Saeed Akram
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Prove:

1. Prove the Demorgans Identities.
2. Let $f : A \rightarrow B$ be a map. Prove that
 - 1) The map f is injective if and only if f has a left inverse.
 - 2) The map f is surjective if and only if f has a right inverse.
 - 3) The map f is a bijection if and only if there exists $g : B \rightarrow A$ such that $f \circ g$ is the identity map on B and $g \circ f$ is the identity map on A .
 - 4) If A and B are finite sets with the same number of elements (i.e., $|A| = |B|$), then $f : A \rightarrow B$ is bijective if and only if f is injective if and only if f is surjective.
3. Let A be a nonempty set. Prove the following:
 - (1) If \sim defines an equivalence relation on A then the set of equivalence classes of \sim form a partition of A .
 - (2) If $\{A_i \mid i \in I\}$ is a partition of A then there is an equivalence relation on A whose equivalence classes are precisely the sets $A_i, i \in I$.
4. Let $f : A \rightarrow B$ be a map. Let $\{A_i\}_{i=1}^{\infty}$ and $\{B_i\}_{i=1}^{\infty}$ be the families of subsets of A and B , respectively. Prove that
 - (a) $f^{-1}(\cap_{i=1}^{\infty} B_i) = \cap_{i=1}^{\infty} f^{-1}(B_i)$.
 - (b) $f^{-1}(\cup_{i=1}^{\infty} B_i) = \cup_{i=1}^{\infty} f^{-1}(B_i)$
 - (c) $f^{-1}(B_1 \sim B_2) = f^{-1}(B_1) \sim f^{-1}(B_2)$.
 - (d) $f(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} f(A_i)$
5. Prove that the Lebesgue outer measure of an interval is its length.
6. Prove that the Lebesgue outer measure is translation invariant, that is, for any set A and number $y, |A + y| = |A|$.
7. Prove that the Lebesgue outer measure is countably subadditive, that is, if $\{E_k\}_k^{\infty}$ is any countable collection of sets, disjoint or not, then $|\cup_{k=1}^{\infty} E_k| \leq \sum_{k=1}^{\infty} |E_k|$.
8. Ch2: Q.5,6,7,8,9
9. Prove that any set of Lebesgue outer measure zero is measurable. In particular, any countable set is measurable.
10. Prove that the union of a finite collection of measurable sets is measurable.
11. Let A be any set and $\{E_k\}_{k=1}^n$ a finite disjoint collection of measurable sets. Then prove that

$$m^*(A \cap [\cup_{k=1}^n E_k]) = \sum_{k=1}^n m^*(A \cap E_k).$$

In particular,

$$m^*(\cup_{k=1}^n E_k) = \sum_{k=1}^n m^*(E_k).$$

12. Prove that the union of a countable collection of measurable sets is measurable.
13. Prove that every interval is measurable.
14. Prove that the collection \mathcal{M} of measurable sets is a σ -algebra that contains the σ -algebra \mathcal{B} of Borel sets. Each interval, each open set, each closed set, each G_δ set, and each F_σ set is measurable.
15. Prove that the translate of a measurable set is measurable.
16. Ch.2: Q.11,12,14,15
17. Ch.2: Q.16,19