COMSATS Institute of Information Technology, Islamabad
Department of Mathematics
Excercise Set 2

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Let $\mathcal{M}$ denotes the $\sigma-$algebra of measurable sets in $\mathbb{R}$. Prove:

1. Let the function f be a real-valued function defined on a measurable set E. Then f is measurable if and only if for each open set O, the inverse image of O under f, $f^{-1}(O) = \{x \in E \mid f(x) \in O\}$, is measurable.

2. Let g be a measurable real-valued function defined on E and f a continuous real-valued function defined on all of $\mathbb{R}$. Then the composition $f \circ g$ is a measurable function on E.

3. Show that the following functions are measurable on their respective domains.

   (a) $f(x) = \begin{cases} x^2, & x < 1 \\ 2, & x = 1 \\ 2 - x, & x > 1. \end{cases}$,

   (b) $f(x) = \dfrac{1}{x}, \; 0 < x < 1.$

   (c) $f(x) = \begin{cases} \dfrac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0. \end{cases}$

   (d) $f(x) = \begin{cases} \tan x, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \infty, & x = \pm\frac{\pi}{2}. \end{cases}$

   (e) $f(x) = \begin{cases} 1, & x \text{ rational number in } [0,1] \\ 0, & x \text{ irrational number in } [0,1]. \end{cases}$

4. Let N be a nonmeasurable subset of $[0,1]$ and define $f(x) = \begin{cases} 1, & x \in N \\ -1, & x \in [0,1] \sim N. \end{cases}$ Show f is a nonmeasurable function on $[0,1]$.

5. Calculate $f^+, f^-, f^+ - f^-, f^+ + f^-$:

   (a) $f(x) = \begin{cases} \infty, & x = \frac{-\pi}{2} \\ \tan x, & \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \infty, & x = \frac{\pi}{2}. \end{cases}$

   (b) $f(x) = \begin{cases} x^2 - 1, & x < 1 \\ 3 - x, & x \geq 1. \end{cases}$

   (c) f(x) = sin(x).

6. Determine if the following functions are "simple" on $[0,1]$:

   (a) $\varphi(x) = \begin{cases} 1, & x = 1/n, n = 1, 2, \ldots \\ -1, & x \neq 1/n. \end{cases}$

(b) $\varphi(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}. \end{cases}$

(c) $\varphi(x) = [x]$ (the greatest integer function).

7. For a sequence $\{f_n\}$ of measurable functions with common domain E, show that each of the following functions is measurable:
$\inf\{f_n\}$, $\sup\{f_n\}$, $\liminf\{f_n\}$ and $\limsup\{f_n\}$.