Abel's test, Drichlet's test
MTH322, Real Analysis II

Lecture # 5 *

September 30, 2015

Recall from the last lecture,

Theorem (Weierstrass M-test). *Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose*

$$|f_n(x)| \leq M_n \qquad (x \in E,\ n = 1, 2, 3, \ldots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Recall from the course Real Analysis I,

Theorem 1 (Theorem 3.28, 1). *$\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

Example 1. *(Application of M-test)*
Consider the series

$$\sum \frac{\sin nx}{n^2},\ x \in \mathbb{R}.$$

Here, $f_n(x) = \frac{\sin nx}{n^2}$. Since the maximum value of $\sin nx$ is 1, $|\frac{\sin nx}{n^2}| \leq \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges by Theorem 1. By the Weierstrass's M-test, we conclude that $\sum \frac{\sin nx}{n^2}$ converges uniformly on $\mathbb{R}$.

The following simple lemma is needed for next two tests.

Lemma 1. *Let $\sum a_n = a$ be the convergent number series with $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=n+1}^{\infty} a_i$. If $\{b_n\}$ is any sequence, then*

(i) $a_n b_n = s_n[b_n - b_{n+1}] - s_{n-1} b_n + s_n b_{n+1}$;

*This lecture is loosely based on the books:

1. Walter Rudin, Principles of Mathematical Analysis, Third Edition
2. Introduction to Mathematical Analysis, First Edition.
3. Gaskil, Narayanaswami, Elements of Real Analysis, First Edition.
4. Bartle, Sherbert, Introduction to Real Analysis, Third Edition
5. Apostol, Mathematical Analysis, Second Edition.

(ii) $a_n b_n = -t_n[b_n - b_{n+1}] + t_{n-1}b_n - t_n b_{n+1}$.

Proof. (i) follows by straightforward computation using $s_n = \sum_{i=1}^{n} a_i$.
(ii) follows from (i) by substituting $t_n = a - s_n$. □

Definition. *We say that* f_n *is* uniformly bounded *on* E *if there exists a number* M *such that*

$$|f_n(x)| < M \qquad (x \in E,\ n = 1, 2, 3, \dots).$$

Theorem 2 (Abel's test for uniform convergence)**.** *If* $\sum f_n$ *converges uniformly on* E*,* $\{g_n\}$ *is uniformly bounded sequence on* E *such that for each* $x \in E$*,* $\{g_n(x)\}$ *is a monotone (increasing or decreasing) sequence, then the series* $\sum f_n g_n$ *converges uniformly on* E*.*

Proof. If $s_n(x)$ denotes the partial sum of $\sum f_n(x)$ and $t_n(x) = \sum_{i=n}^{\infty} f_i(x)$. Using identity (ii) of Lemma 1, we obtain

$$\sum_{i=n}^{m} f_i(x)g_i(x) = \sum_{i=n}^{m}[-t_i(x)(g_i(x) - g_{i+1}(x))] + t_{n-1}(x)g_n(x) - t_m(x)g_{m+1}(x).$$

Therefore, if M is a uniform bound for $\{g_n\}$ then for any $\epsilon > 0$, there is an integer N such that $m \geq n \geq N$, $x \in E$ implies

$$\begin{array}{rcl}
|\sum_{i=n}^{m} f_i(x)g_i(x)| & \leq & \sum_{i=n}^{m}[|t_i(x)||g_i(x) - g_{i+1}(x)|] + |t_{n-1}(x)||g_n(x)| + |t_m(x)||g_{m+1}(x)| \\
& \leq & \sum_{i=n}^{m}[|t_i(x)||g_i(x) - g_{i+1}(x)|] + |t_{n-1}(x)|M + |t_m|M \\
& \leq & \frac{\epsilon}{4M}\sum_{i=n}^{m}|g_i(x) - g_{i+1}(x)| + \frac{\epsilon}{4M}M + \frac{\epsilon}{4M}M \text{ (since } \sum f_n \text{ is converges uniformly)} \\
& \leq & \frac{\epsilon}{4M}|g_n(x) - g_{m+1}(x)| + \frac{\epsilon}{4M}M + \frac{\epsilon}{4M}M = \epsilon \qquad \text{(since } \{g_n\} \text{ is monotne)} \\
& \leq & \frac{\epsilon}{4M}2M + \frac{\epsilon}{4M}M + \frac{\epsilon}{4M}M = \epsilon, \qquad \text{(since } \{g_n\} \text{ uniformly bounded)}
\end{array}$$

the uniform convergence of $\sum f_n g_n$ now follows by Corollary **??**. □

Example 2. *(Application of Abel's test)*
Consider the series

$$\sum \frac{x^n}{n^2(1 - x^{2n})}, \qquad x \in (0, 1).$$

Fix r *in* $(0, 1)$*. Let* $f_n(x) = \frac{x^n}{n^2}$ *and* $g_n = \frac{1}{1-x^{2n}}$*.*
The series $\sum f_n = \sum \frac{x^n}{n^2}$ *converges uniformly on* $[-r, r]$ *by Weierstrass's M-test.*
The sequence $\{g_n\}$ *is monotone decreasing and bounded below.*
(To prove this, note that $0 < (1 - x^{2n}) \leq (1 - x^{2n+2}) \leq 1$ *for all* $x \in [-r, r]$*. Thus, for such* x*,* $1 \leq \frac{1}{1-x^{2n+2}} \leq \frac{1}{1-x^{2n}}$*).*
Consequently, Abel's test applies and our series converges uniformly on $[-r, r]$*.*

Theorem 3 (Dirichlet's test for uniform convergence)**.** *Given the series* $\sum f_n$ *with* s_n *uniformly bounded on* E*. If* $\{g_n\}$ *is a monotone (increasing or decreasing) sequence of functions which converges uniformly to* 0 *on* E*, then the series* $\sum f_n g_n$ *converges uniformly on* E*.*

Proof. If $s_n(x)$ denotes the partial sum of $\sum f_n(x)$ then using identity (i) of Lemma 1, we obtain

$$\sum_{i=n}^{m} f_i(x)g_i(x) = \sum_{i=n}^{m}[s_i(x)(g_i(x) - g_{i+1}(x))] - s_{n-1}(x)g_n(x) + s_m(x)g_{m+1}(x).$$

Therefore, if M is a uniform bound for $\{s_n\}$ then for any $\epsilon > 0$, there is an integer N such that $m \geq n \geq N$, $x \in E$ implies

$$\begin{aligned}
|\textstyle\sum_{i=n}^{m} f_i(x)g_i(x)| &\leq \textstyle\sum_{i=n}^{m}[|s_i(x)||g_i(x) - g_{i+1}(x)|] + |s_{n-1}(x)||g_n(x)| + |s_m(x)||g_{m+1}(x)| \\
&\leq M\textstyle\sum_{i=n}^{m}|g_i(x) - g_{i+1}(x)| + M|g_n(x)| + M|g_{m+1}| \\
&\leq M\textstyle\sum_{i=n}^{m}|g_i(x) - g_{i+1}(x)| + M\frac{\epsilon}{4M} + M\frac{\epsilon}{4M} \qquad (\because \{g_n\} \text{ converges uniformly }) \\
&\leq M|g_n(x) - g_{m+1}(x)| + \frac{\epsilon}{2} \qquad (\because \{g_n\} \text{ is monotone }) \\
&\leq M\frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon, \qquad (\because \{g_n\} \text{ converges uniformly to } 0 \text{ })
\end{aligned}$$

the uniform convergence of $\sum f_n g_n$ now follows by Corollary **??**. □

Theorem 4. *(Theorem 8.30, 5) For any real $x \neq 2m\pi$ (m is an integer), we have*

$$\sum_{k=1}^{n} e^{ikx} = e^{ix}\frac{1 - e^{inx}}{1 - e^{ix}} = \frac{\sin(nx/2)}{\sin(x/2)}e^{i(n+1)x/2}.$$

This identity yields the following estimate:

$$|\sum_{k=1}^{n} e^{ikx}| \leq \frac{1}{|\sin(x/2)|}.$$

Example 3. *(Application of Dirichlet's test)*
Consider the series

$$\sum_{n=1}^{\infty}\frac{e^{inx}}{n^p}, \qquad p \in (0,\infty) \text{ and } x \in [a,b] \text{ such that } 0 < a < b < 2\pi.$$

Let $f_n(x) = e^{ikx}$ and $g_n(x) = 1/n^p$. The conditions required by the Dirichlet's test are met. First, by Theorem 4, for any $x \in [a,b]$, we have

$$\begin{aligned}
|s_n(x)| &= |\textstyle\sum_{k=1}^{n} e^{ikx}| \leq \frac{1}{\sin(x/2)} \\
&\leq \frac{1}{m},
\end{aligned}$$

where $m = \max\{\sin\ (a/2),\ \sin\ (b/2)\}$. Thus $\{s_n\}$ is uniformly bounded by $M = 1/m$ on $[a,b]$. Second, the sequence $\{g_n(x)\}$, free of the variable x, is a positive sequence that converges monotonically and uniformly to 0 on $[a,b]$. Dirichlet's test implies that $\sum_{n=1}^{\infty}\frac{e^{ikx}}{n^p}$ converges uniformly on $[a,b]$. Notice that neither Weierstrass, M-test nor Abel's test applies in any obvious way to help us resolve the question of uniform convergence of this aeries.

Exercises

Prove the following statements:

1. Let $\sum f_n$ converges pointwise/uniform on E. Then the limit, f, is unique.

2. Let $\sum f_n$ and $\sum g_n$ be two infinite series that are uniformly convergent on E. If f and g are the respective limits, and a, b are two real constants, then

$$\sum (af_n \pm bg_n) = af \pm bg \quad \text{uniformly on } E.$$

   What can be said about the uniform convergence of $\sum f_n \cdot g_n$ and $\sum |f_n|$ on E?

3. Theorem 4.

4. Discuss the uniform convergence of the series

   i $\sum_{n=1}^{\infty} \frac{\sin\ nx}{n^p}$, $\quad p \in (0, \infty)$ and $x \in [a, b]$ such that $0 < a < b < 2\pi$.

   ii $\sum_{n=1}^{\infty} \frac{\cos\ nx}{n^p}$, $\quad p \in (0, \infty)$ and $x \in [a, b]$ such that $0 < a < b < 2\pi$.